Convolution revisited

That probability theory can serve as an invaluable tool in solving engineering problems is amply demonstrated by the use of convolution techniques providing solutions previously attainable by transform methods

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Few mathematical operations are more important to the engineer than convolution and transform analysis. In this article, the operation of convolution is explored—starting with discrete rather than continuous convolution because of the relative ease of comprehension involved. With this foundation, the study is extended to continuous convolution. A proof of the convolution theorem will show that convolution and transform analysis are closely related. Of much more interest, however, is an intuitive explanation of why convolution and transform analysis techniques lead to exactly the same solution of a given problem. Perhaps the two most important applications of convolution deal with the analysis of linear systems and the sums of independent random variables—the latter problem being used to introduce discrete convolution.

Adding random variables

Transformers of a certain type are delivered by two companies, which will be called A and B, in lots of three and four, respectively. In a lot from company A, there will be zero, one, two, or three defective transformers. The probability that each of these numbers of defects, represented by a ($a = 0, 1, 2, 3$), will occur is given as

a	$P(a)$
0	0.4
1	0.3
2	0.2
3	0.1

Similarly, the probability of b ($b = 0, 1, 2, 3, 4$) occurrences is

b	$P(b)$
0	0.3
1	0.2
2	0.2
3	0.2
4	0.1

in a shipment from company B. The problem is to find the probabilities of the total number of defects in two shipments, one from each company. This sum or total number of defects will be indicated as c ($c = 0, 1, 2, 3, 4, 5, 6, 7$).

The probability that $c = 0$ is the probability that $a = 0$ and $b = 0$; that is, that there are no defects in either shipment. This is written as

$$P(c = 0) = P(a = 0 \text{ and } b = 0)$$

If the events $a = 0$ and $b = 0$ are independent of each other, which we assume here and which is necessary if the solution is to be a convolution, then this probability reduces to the product

$$\begin{aligned} P(c = 0) &= P(a = 0) \times P(b = 0) \\ &= 0.4 \times 0.3 \\ &= 0.12 \end{aligned}$$

The probability that we have a total of one defect is the probability that the shipment from A has one defect and the shipment from B none, or vice versa. That is,

$$P(c = 1) = P[(a = 1 \text{ and } b = 0) \text{ or } (a = 0 \text{ and } b = 1)]$$

Using an axiom of probability theory, the probability of the "or" statement within the brackets is changed to the sum of two probabilities.

$$\begin{aligned} P(c = 1) &= P(a = 1 \text{ and } b = 0) + P(a = 0 \text{ and } b = 1) \\ &= P(a = 1)P(b = 0) + P(a = 0)P(b = 1) \\ &= 0.3 \times 0.3 + 0.4 \times 0.2 \\ &= 0.17 \end{aligned}$$

Similarly,

$$P(c = 2) = P[(a = 2 \text{ and } b = 0) \text{ or } (a = 1 \text{ and } b = 1) \text{ or } (a = 0 \text{ and } b = 2)]$$

$$+ P(a = 1)P(b = 1)$$

$$+ P(a = 0)P(b = 2)$$

$$= 0.2 \times 0.3 + 0.3 \times 0.2 + 0.4 \times 0.2$$

$$= 0.20$$

Continuing in this way, one can obtain the probabilities for all eight possible values of c. The result is given as

c	$P(c)$
0	0.12
1	0.17
2	0.20
3	0.21
4	0.16
5	0.09
6	0.04
7	0.01

Discrete convolution

Although it is possible to solve this problem in the manner just described, it is highly desirable to find a shortcut to determine the probabilities of the values of c. Notice how the entries for a and b are used to find those for c. The first entry for c is the product of the first entries for a and b. The second entry for c is entry 1 of b times entry 2 of a plus entry 2 of a times entry 1 of b. The third entry of c is the sum of cross terms 1 and 3, 2 and 2, 3 and 1 of a and b.

We can systematize the process of finding the entries for c in the following way. Write the probabilities for a and b as sequences A and B:

$$A = [0.4\ 0.3\ 0.2\ 0.1]$$

$$B = [0.3\ 0.2\ 0.2\ 0.2\ 0.1]$$

... sequences, say B. Call the reversed sequence B_{inv}

$$B_{inv} = [0.1\ 0.2\ 0.2\ 0.2\ 0.3]$$

Position sequence A and the inverted sequence B_{inv} so that the first right-hand term of B_{inv} is under the first left-hand term of A:

```
            0.4 0.3 0.2 0.1
0.1 0.2 0.2 0.2 0.3
```

The probability of 0 defects is the product of the overlapping numbers 0.4 and 0.3. Now shift the inverted sequence *one* position to the right:

```
        0.4 0.3 0.2 0.1
0.1 0.2 0.2 0.2 0.3
```

The probability of *one* defect is the sum of the overlapping products 0.2×0.4 and 0.3×0.3. The remaining terms in c are obtained by shifting the inverted sequence one step at a time to the right, and for each step summing the overlap products.

The process of inverting a sequence, sliding it one step at a time to the right, and summing the overlap products is called discrete convolution. It is sometimes called serial multiplication. (An excellent discussion of serial multiplication or discrete convolution is given by Bracewell.[1])

The asterisk (*) is generally used to indicate discrete as well as continuous convolution. Thus, we write

$$C = A * B \tag{1}$$

$$= [0.4\ 0.3\ 0.2\ 0.1] * [0.3\ 0.2\ 0.2\ 0.2\ 0.1]$$

$$= [0.12\ 0.17\ 0.20\ 0.21\ 0.16\ 0.09\ 0.04\ 0.01]$$

Before proceeding with the theoretical development, the reader who is not familiar with discrete convolution should spend some time practicing the technique. The objective of this practice is to get a feeling

This box contains a number of exercises intended to acquaint the reader with both the techniques and results of discrete convolution. The reader is encouraged to look for interesting forms or combinations and perhaps deduce some of the properties of convolution that have not been discussed here.

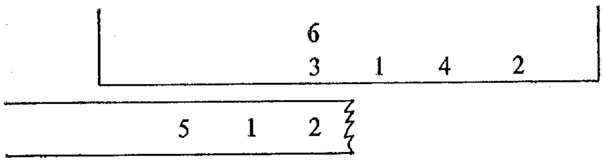
(1) [3 1 5] * [4 4 7] = [12 16 45 27 35]
(2) [1 4 4 1] * [2 6 2] = [2 14 34 34 14 2]
(3) [1 3 3 1] * [1 2 1] = [1 5 10 10 5 1]
(4) [9 5 1] * [5 3 1] = [45 52 29 8 1]
(5) [4 1 7 8] * [· · · 0 0 1 0 0 0 · · ·] = [· · · 0 0 4 1 7 8 0 0 0 · · ·]
(6) [1 1 1 1] * [1 1 1 1] = [1 2 3 4 3 2 1]
(7) [8 1 6 3 2] * [4 1 5] = [32 12 65 23 41 17 10]

for the operation. This will facilitate understanding of further developments.

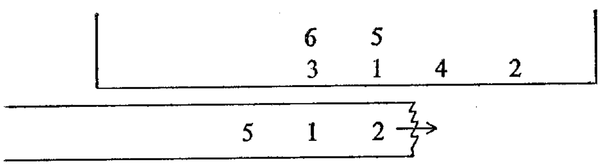
There are a number of ways one may carry out the mechanics of the operation. It is well to start with a procedure that is a little tedious but very instructive. Consider the convolution of two sequences, A and B, defined as

$$A = [3\ 1\ 4\ 2] \qquad B = [2\ 1\ 5]$$

Write the sequence A on one piece of paper. On the edge of a second piece of paper, write the inverted sequence B_{inv} with the same spacing as A.



Slide the B_{inv} sequence paper to the right until the 2 is under the 3. Obtain the product 6 and write it above the 3. Shift the B_{inv} sequence paper one step to the right:



Add the paired products 2×1 and 1×3 to obtain 5; write 5 above the 1. Repeat this operation until the paired products are finished; one should obtain

$$[3\ 1\ 4\ 2] * [2\ 1\ 5] = [6\ 5\ 24\ 13\ 22\ 10]$$

A number of examples are given in the tinted box on page 88. The reader should work one or two with the method just outlined; once he has a feeling for what is happening he will want to find a simpler method. The following approach seems to be as simple as possible. Write out the sequences in a conventional multiplication format. Do not invert either sequence. Use the conventional multiplication procedure with the exception of carrying tens—do not carry tens.

$$\begin{array}{cccccc}
 & & 3 & 1 & 4 & 2 \\
 & & & 2 & 1 & 5 \\
\hline
 & & 15 & 5 & 20 & 10 \\
 & 3 & 1 & 4 & 2 & \\
6 & 2 & 8 & 4 & & \\
\hline
6 & 5 & 24 & 13 & 22 & 10
\end{array}$$

If one uses this technique to obtain some of the results in numerical examples that are given, it will be easier to understand, and to check, the following properties or facts. No rigorous proofs will be given here.

Consider the convolution of two sequences A and B to obtain a third sequence C.

$$A = [a_0\ a_1 \cdots a_j \cdots]$$

$$B = [b_0\ b_1 \cdots b_j \cdots]$$

$$C = [c_0\ c_1 \cdots c_i \cdots]$$

1. The elements of sequence C can be expressed as

$$c_i = \sum_{j=0}^{i} a_j b_{i-j} \tag{2}$$

2. If the sequence A has n_A terms and the sequence B has n_B terms, the number of terms in sequence C is

$$n_C = n_A + n_B - 1 \tag{3}$$

3. Convolution is commutative:

$$A * B = B * A \tag{4}$$

This means that either sequence can be inverted and shifted.

4. The product of the sum of the elements in sequence A and the sum of the elements in sequence B is equal to the sum of the elements in sequence C. That is,

$$\left(\sum_{i=0}^{n_A-1} a_i\right)\left(\sum_{j=0}^{n_B-1} b_j\right) = \sum_{k=0}^{n_A+n_B-1} c_k \tag{5}$$

This property should always be used as a check on simple discrete convolution problems. It is a particularly significant property in the sums of random variables problem since the three sequences must each add up to one. This is true because the probabilities in an experiment always total one.

We will see the continuous convolution analogs to these four properties in the next section.

Before moving on to a discussion of continuous convolution, the problem of inverting the discrete convolution process will be described. Consider the following numerical example:

$$[3\ 1\ 4\ 2] * [b_0\ b_1\ b_2] = [6\ 5\ 24\ 13\ 22\ 10]$$

In the inversion problem, the task is to find the sequence B. An algorithm for generating the elements of B is easily established. Consider the following:

$$\begin{array}{cccccc}
 & & 6 & 5 & 24 & 13\ 22\ 10 \\
 & & 3 & 1 & 4 & 2 \\
b_2 & b_1 & b_0 & & &
\end{array}$$

Since $3 \times b_0 = 6$, b_0 is obviously 2. Now shift B to the right one step. We find b_1 from the expression

$$3 \times b_1 + 1 \times b_0 = 5$$

But b_0 is known, so we can write

$$b_1 = \frac{5 - 1 \times b_0}{3} = 1$$

The general expression for b_i is

$$b_i = \frac{c_i - \sum_{j=0}^{i-1} b_j a_{i-j}}{a_0} \tag{6}$$

Although this expression is quite simple and straightforward in principle, there are serious problems concerned with applying it. If the measured values of C are noisy, that is, if their measured values do not equal their true values, then the values of b_i obtained from Eq. (6) will be increasingly inaccurate as i increases. This is true because the errors accumulate in the summation of $b_j a_{i-j}$ in the equation. An interesting example of this accumulation of error is given by Landgrebe.[2]

At this point, our discussion of discrete convolution will end in order to review the more familiar concepts of continuous convolution. We will return to discrete convolution after borrowing an idea or two from continuous convolution.

Continuous convolution

This section will be essentially a review for many readers. However, it would be well for anyone who does not have a clear understanding of the concept of continuous convolution to follow the ideas closely. He should particularly try to relate each concept in continuous convolution to its discrete-convolution analog.

The convolution of two continuous functions $f(x)$ and $g(x)$ is written as

$$h(x) = \int_{-\infty}^{\infty} f(\lambda) g(x - \lambda)\, d\lambda \tag{7}$$

(The reader may be used to seeing limits of 0 to x on the integral. We shall see later that this is true only for functions that are zero for x less than zero.) Equation (7) is analogous to Eq. (2). Note that the dummy variable of integration λ in (7) plays a role similar to the dummy variable of summation j in (2). If $f(x)$ and $g(x)$ are probability density functions of two continuous random variables, then $h(x)$ is the probability density function of their sum.

If the variable substitution $x - \lambda = \beta$ in Eq. (7), we obtain

$$h(x) = \int_{-\infty}^{\infty} f(x - \beta) g(\beta)\, d\beta \tag{8}$$

which tells us that

$$f(x) * g(x) = g(x) * f(x) \tag{9}$$

This result is of course analogous to Eq. (4).

Property 4 of the previous section also has an analogy in continuous convolution—the product of the area under the $f(x)$ and $g(x)$ curves equals the area under the $h(x)$ curve. This result is obtained by integrating both sides of Eq. (7) over all x:

$$\begin{aligned}\int_{-\infty}^{\infty} h(x)\, dx &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\lambda) g(x - \lambda)\, d\lambda\, dx \\ &= \int_{\lambda=-\infty}^{\infty} f(\lambda) \int_{-\infty}^{\infty} g(x - \lambda)\, dx\, d\lambda \\ &= \int_{-\infty}^{\infty} f(x)\, dx \int_{-\infty}^{\infty} g(x)\, dx \end{aligned} \tag{10}$$

The last step is possible because the integral over all x of $g(x - \lambda)$ is equal to the integral of $g(x)$.

We turn now to a graphical interpretation of the process of convolution. Consider the functions $f(x)$ and $g(x)$ in Fig. 1.

The minus sign in the term $g(x - \lambda)$ in Eq. (7) represents a reversal of the order of the values of $g(\lambda)$; that is, $g(-\lambda)$ is the mirror image of $g(\lambda)$ with respect to the g axis. This represents a folding or convolving of $g(x)$. The x in $g(x - \lambda)$ represents shifting of the folded function. This process is illustrated in

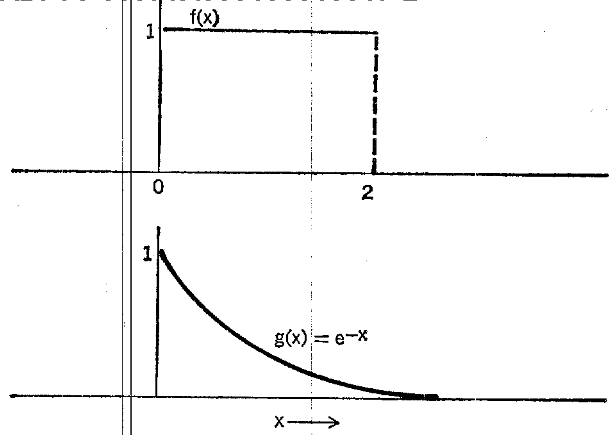


FIGURE 1. Graphical representation of the functions described by f(x) and g(x).

FIGURE 2. Description of the process of convolution for continuous functions.

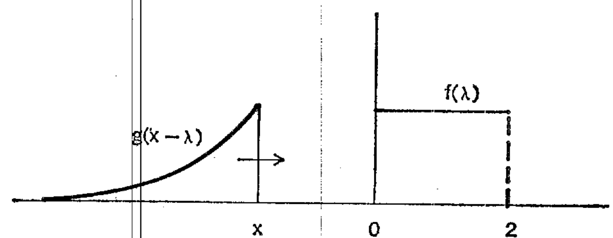


Fig. 2. For this case, where $g(x)$ has zero value for negative x, x is easily identified as the position of the forward edge of the function $g(x - \lambda)$ as it is shifted to the right.

The graphical interpretation of continuous convolution can now be summarized. The reader should compare this with the discrete convolution operation as previously described. Reverse the order of (or fold) one function and, starting from the far left, shift it to the right one step at a time. (The only difference with continuous convolution is that there are an infinite number of infinitesimal steps. Each position of the shift represents a value of the continous variable x.) For each position x, find the integral of the product of the functions $f(\lambda)$ and $g(x - \lambda)$ for all λ. This last step is of course analogous to finding the sum of the paired terms in discrete convolution.

An integral closely related to the convolution integral is the correlation integral:

$$\int_{-\infty}^{\infty} f(\lambda) g(x + \lambda)\, d\lambda$$

This integral is used in describing signals where $f(\cdot)$ and $g(\cdot)$ are the same function, and describing the interrelation of signals when they are different. It also gives the probability density function of the *difference* of two continuous independent random variables.

One of the most challenging problems relating to

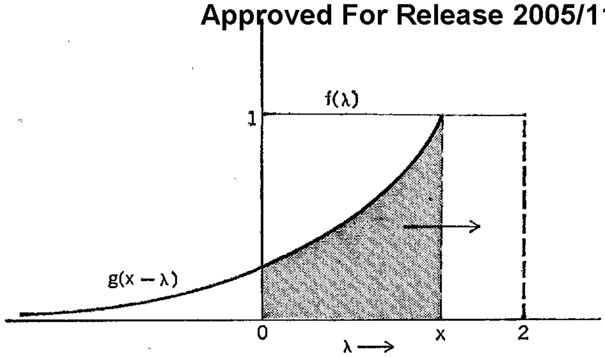
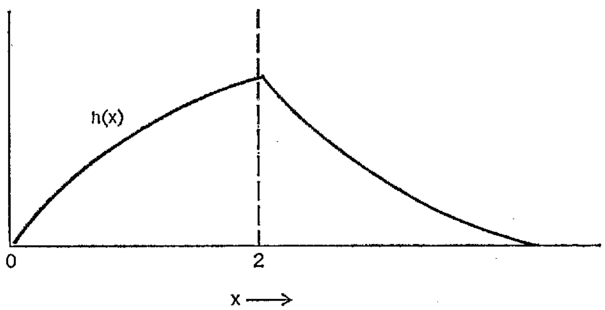


FIGURE 3. Determination of the limits of integration for the convolution integral.

FIGURE 4. Exponential impulse response to a square wave of a circuit.



application of the convolution integral is that of setting the limits on the integral. The $-\infty$ to $+\infty$ limits of Eqs. (7) and (8) apply only if the functions $f(x)$ and $g(x)$ are nonzero for all x. If we wish to integrate a function such as

$$g(x) = \begin{cases} e^{-x} & 0 \leq x \leq \infty \\ 0 & -\infty \leq x > 0 \end{cases}$$

over all x, we do so by restricting the limits from 0 to ∞; hence, a similar technique is used in the convolution problem.

Consider (Fig. 3) the convolution of the functions shown in Fig. 1, for x between 0 and 2. The area under the product of $f(\lambda)$ and $g(x - \lambda)$ is shown in color. This is the area whose value is found by Eq. (7). It is apparent in this case that the integration must be carried out from 0 to x, rather than from $-\infty$ to ∞.

We would like to have a relatively simple means of finding the limits in the general case. For the example given, the lower limit of the function $g(x - \lambda)$ is $-\infty$ and the lower limit on $f(\lambda)$ is 0. When we integrated, we chose the largest of the two as our lower limit. The upper limit on $g(x - \lambda)$ is x; the upper limit of $f(\lambda)$ is 2. We, in turn, chose the smallest of these for our upper limit of integration. Therefore, the general rule is: Given two functions with lower limits m_1 and m_2 and upper limits M_1 and M_2, use $\max(m_1, m_2)$ to $\min(M_1, M_2)$ as the range of integration.

This rule is of course quite logical. It simply specifies the range of overlap of the two functions. The next task is to find the limits m and M for the two functions. The limits for the fixed function $f(\lambda)$ do not change. They are simply the limits on the original function $f(x)$; that is,

$$m_x \leq \lambda \leq M_x \tag{11}$$

The limits of the sliding function $g(x - \lambda)$ change as x changes. If the original argument of $g(\cdot)$ is called y, and we are given the limits on y as m_y and M_y, then

$$m_y \leq y \leq M_y$$

But $y = x - \lambda$ under the convolution transformation, so

$$m_y \leq x - \lambda \leq M_y$$

$$\therefore \quad x - M_y \leq \lambda \leq x - m_y \tag{12}$$

Since the limits on the fixed function are given by (11), and the limits on the sliding function by (12), the integration or overlap range is

$$\max(m_x, x - M_y) \leq \lambda \leq \min(M_x, x - m_y) \tag{13}$$

It is apparent that the range of integration depends on x, the variable of the resultant of the convolution operation. Using (13) to establish the limits for the convolutions of the functions shown in Fig. 1, one has

$$h(x) = \int_{\max(0,x-\infty)}^{\min(2,x-0)} 1 \times e^{-(x-\lambda)}\, d\lambda \tag{14}$$

Note that the upper limit in (14) is different for $0 \leq x \leq 2$ and $x > 2$. To see why this is reasonable, refer to Fig. 3 and visualize the change in the problem when the sliding function slides past $x = 2$.

Thus, the integral of Eq. (14) has different limits for different ranges of x:

$$h(x) = \begin{cases} \int_0^x e^{-x} e^{\lambda}\, d\lambda & 0 \leq x \leq 2 \\ \int_0^2 e^{-x} e^{\lambda}\, d\lambda & x > 2 \end{cases} = \begin{cases} 1 - e^{-x} & 0 \leq x \leq 2 \\ e^{-x}(e^2 - 1) & x > 2 \end{cases} \tag{15}$$

The result, plotted in Fig. 4, is the response to a square wave of a circuit characterized by an exponential impulse response.

As a second example, consider the general problem of causal time signals applied to linear time-invariant networks, in which a signal has zero value for $t < 0$. In general, the range of both is zero to infinity. Substitution into Eq. (13) yields

$$0 \leq \lambda \leq t \tag{16}$$

The variable t is the time at which the network output is observed and it is, in general, a function of the input applied over the entire time period from zero to t. The output at the instant t is the *effect;* the input signal applied after $t = 0$ is the *cause.* This approach to establishing the limits of integration is,

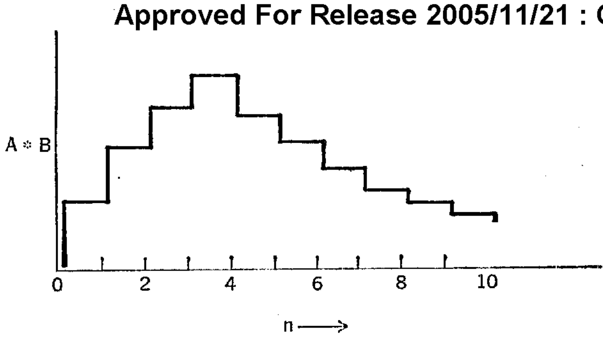


FIGURE 5. Graphical representation of the discrete convolution A * B.

in the author's opinion, the most satisfactory in the long run. It implies and demands an understanding of the problem.

An alternative method, using step functions and a table of special cases, has been proposed by Ross.[3] In order to establish the concept of convolution more clearly, it is instructive at this point to solve a discrete convolution problem that is the analog of the continuous problem that led to Eq. (15). As a parallel to the function $f(\lambda)$, consider a sequence of four identical elements:

$$A = [1\ 1\ 1\ 1]$$

The discrete analogy to a decaying exponential is a sequence in which each element is a constant fraction of the preceding element. That is,

$$b_i = kb_{i-1} \qquad 0 \leq k \leq 1$$

If we let $b_0 = 1$ and $k = 0.8$, we obtain

$$B = [1.00\ 0.80\ 0.64\ 0.51\ 0.41\ 0.33\ 0.26\ 0.21\ 0.17\ 0.13 \cdots]$$

(Note that in this analogy no attempt has been made to make the continuous and discrete problems numerically analogous. Only the shapes are analogous.) Hence,

$$A * B = [1.00\ 1.80\ 2.44\ 2.95\ 2.36\ 1.89\ 1.51\ 1.21\ 0.98\ 0.77 \cdots]$$

A bar graph of A * B is shown in Fig. 5 (compare Fig. 5 with Fig. 4).

More on discrete convolution

In discrete convolution, a problem is encountered in establishing the limits of summation, which is similar to the problem of establishing the limits of integration in continuous convolution. The independent variable or index of $f(k)$ is of course k and is assumed to have a range:

$$r_k \leq k \leq R_k$$

The index of $g(l)$ has a range:

$$r_l \leq l \leq R_l$$

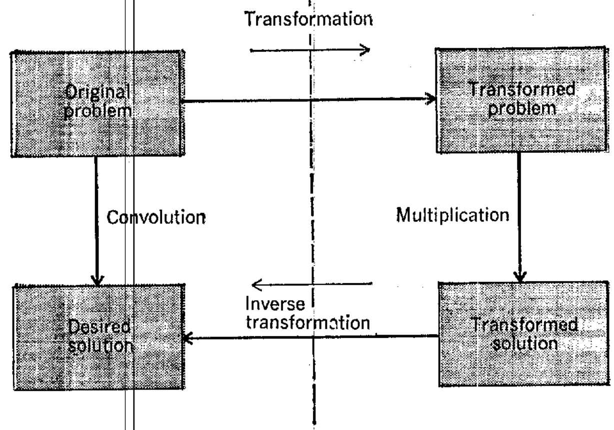


FIGURE 6. Block diagram of the alternate paths of solution that are offered by the techniques involved in convolution and transformation.

In a manner similar to that of the previous section, we obtain for the limits of summation:

$$\max(r_k, n - R_l) \leq k \leq \min(R_k, n - r_l) \tag{17}$$

The two forms of convolution can now be written for comparison:

$$h(x) = \int_{\max(m_x, x-M_y)}^{\min(M_x, x-m_y)} f(\lambda)g(x - \lambda)\, d\lambda \tag{18}$$

$$h(n) = \sum_{k=\max(r_k, n-R_l)}^{\min(R_k, n-r_l)} f(k)g(n - k) \tag{19}$$

If our discrete functions are "causal" ($k = 0, 1, 2, \ldots$ and $l = 0, 1, 2, \ldots$), then

$$h(n) = \sum_{k=0}^{n} f(k)g(n - k) \tag{20}$$

We proceed now to the subject of transform analysis and its relation to convolution.

Transform analysis

The types of problems that can be solved using convolution techniques can also be solved using various transform techniques (e.g., Laplace, Fourier, Z, etc.). Figure 6 shows a simple block diagram of alternate solution paths for a given problem. This diagram applies to the convolution problem, and is applicable to the transform solution of problems in differential equations. In probability theory, the transform—or a very close relative of it—is called the characteristic function.

There are two basic reasons for using transform techniques. The first is to simplify the mathematics required to reach the desired solution. The second is to obtain an insight or understanding of a problem that is unavailable without transform techniques. Both of these reasons apply in this study.

Let us start by proving that we obtain the same solution by following either the transform or convolution path in Fig. 6. This involves the proof of the convolution theorem. We will prove the theorem for the Fourier transform; a similar proof holds for other transforms. The Fourier transform of the function $f(x)$ is

$$F(s) = \int_{-\infty}^{\infty} f(x)e^{-i2\pi xs}\,dx \tag{21}$$

Using this definition, we take the transform of both sides of Eq. (7).

$$H(s) = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\lambda)g(x-\lambda)\,d\lambda\right]e^{-i2\pi xs}\,ds$$

$$= \int_{-\infty}^{\infty} f(\lambda)\left[\int_{-\infty}^{\infty} g(x-\lambda)e^{-i2\pi xs}\,dx\right]d\lambda$$

Letting $x - \lambda = y$,

$$H(s) = \int_{-\infty}^{\infty} f(\lambda)e^{-i2\pi\lambda s}\left[\int_{-\infty}^{\infty} g(y)e^{-i2\pi ys}\,dy\right]d\lambda$$

$$= \int_{-\infty}^{\infty} f(\lambda)e^{-i2\pi\lambda s}[G(s)]\,d\lambda$$

$$= F(s)\cdot G(s) \tag{22}$$

This result establishes that convolution in the plane of the original variable is equivalent to multiplication in the plane of the transform variable.

If the variable has discrete values, the Fourier transform is

$$F(s) = \sum_{\text{all } k} f(k)e^{-i2\pi ks} \tag{23}$$

We now repeat the proof of the convolution theorem for discrete variables.

$$H(s) = \sum_{n}\left[\sum_{\lambda} f(\lambda)g(n-\lambda)\right]e^{-i2\pi ns}$$

$$= \sum_{\lambda} f(\lambda)\left[\sum_{n} g(n-\lambda)e^{-i2\pi ns}\right]$$

Letting $n - \lambda = m$

$$H(s) = \sum_{\lambda} f(\lambda)e^{-i2\pi\lambda s}\left[\sum_{m} g(m)e^{-i2\pi ms}\right]$$

$$= \sum_{\lambda} f(\lambda)e^{-i2\pi\lambda s}[G(s)]$$

$$= F(s)\cdot G(s) \tag{24}$$

These two parallel proofs are quite important, since they establish the validity of the use of transforms to solve convolution problems. They are not, however, particularly satisfying since they tend not to answer the question of *why* convolution and transform techniques lead to the same solution. We shall try to provide the reader with an answer to that question in the following development.

Discrete functions will be employed since the point to be made is easily seen in that case. Assume that $f(k)$ and $g(k)$ are discrete functions that can be written in the form of sequences, as

$$f(k) = [f_0\ f_1\ f_2\ \cdots] \qquad k = 0, 1, 2, \cdots$$

$$g(k) = [g_0\ g_1\ g_2\ \cdots] \qquad k = 0, 1, 2, \cdots$$

From Eq. (23), the transforms of $f(k)$ and $g(k)$ are

$$F(s) = f_0e^{-i2\pi 0s} + f_1e^{-i2\pi 1s} + f_2e^{-i2\pi 2s} + \cdots$$

$$G(s) = g_0e^{-i2\pi 0s} + g_1e^{-i2\pi 1s} + g_2e^{-i2\pi 2s} + \cdots$$

To find $f(k) * g(k)$, we can use the convolution theorem [Eq. (24)] just established. Multiply $F(s)$ and $G(s)$, and collect terms with the same exponent:

$$H(s) = f_0g_0e^{-i2\pi 0s} + (f_0g_1 + f_1g_0)e^{-i2\pi 1s}$$

$$+ (f_0g_2 + f_1g_1 + f_2g_0)e^{-i2\pi 2s} + \cdots \tag{25}$$

The convolution of $f(k)$ and $g(k)$ is obtained by taking the inverse transform of $H(s)$; that is, by finding the function $h(k)$ that when transformed yields $H(s)$. Inspection of Eqs. (23) and (25) suggests that this function is

$$h(k) = [(f_0g_0)\ (f_0g_1 + f_1g_0)\ (f_0g_2 + f_1g_1 + f_2g_0)\ \cdots]$$

We obtain this result by noting the power of the exponent with which each term is associated. What is happening here should now be apparent. The transform operation is simply performing a bookkeeping function. The transform process associates the correct coefficients with the correct power terms. Multiplication of the transforms leads to the same association of coefficients as does convolution. Thus, convolution and the transform process are seen as two essentially parallel methods of keeping track of sets of coefficients.

This argument is essentially applicable to the continuous case, though it is by no means as easily demonstrated.

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Scanning the issues

Advance tables of contents
Translated journals
Special publications

Conscience of the City. In an essay titled "Like It Is in the Alley," a Harvard research psychiatrist, who for many years has been working with urban and rural poor people, transcribes for us what a young boy named Peter has to say about his situation. Peter lives, Robert Coles writes, in the heart of what we in contemporary America have chosen (ironically, so far as history goes) to call an "urban ghetto."

"In the alley it's mostly dark, even if the sun is out. But if you look around, you can find things. I know how to get into every building, except that it's like night once you're inside them, because they don't have lights. So, I stay here. You're better off. It's no good on the street. You can get hurt all the time, one way or the other. And in buildings, like I told you, it's bad in them, too. But here it's o.k. You can find your own corner, and if someone tries to move in you fight him off. We meet here all the time, and figure out what we'll do next. It might be a game, or over for some pool, or a coke or something. You need to have a place to start out from, and that's like it is in the alley; you can always know your buddy will be there, provided it's the right time. So you go there, and you're on your way, man."

Peter's plight—his vivid "you're on your way, man" tells us unmistakably what he needs is a way out—is shared by millions of others, both black and white. The modern city, the traditional place for a man to shake his fetters, has been and continues to be, for many millions, nothing more than a cage. In a zoo, the big cats pace back and forth, back and forth, along the bars, or they lie somnolent. For many, the city is only a zoo for our kind of animal.

What is to be done? For the fact is that the problem of the poor man in the city ghetto is but part of a much larger constellation of problems that have multiplied increasingly in recent years. Whether or not the city is a viable structure in our time has come seriously into question. A measure of just how seriously—beyond the accounts we all share through our newspapers—and a measure of the diversity of questions that can be asked about what may well be unsolvable problems comes to us in a volume of papers, of which Coles' "Like It Is in the Alley" is one.

The volume is a special issue of *Dædalus,* the journal of the American Academy of Arts and Sciences, and is an outgrowth of an AAAS study of the future of urbanism. At a time when private industries—and especially those organizations that are built around modern engineering—are being beseeched to contribute to the solution of urban problems, there is no question that engineers must take a much wider compass on such problems than their own specialized literature (including the best of their vaunted systems engineering) has thus far provided. This volume of *Dædalus,* entitled "The Conscience of the City," is earnestly recommended for your attention, whether you are working directly on urban problems or not. These essays reorient, they provoke, they stimulate, they temper one's hopes, they inform, they unsettle, and they startle, and, as in Coles' case—which weighs like the center of gravity through the direct individual human plight—they break your heart.

They reorient almost immediately in Melvin M. Webber's paper, "The Post-City Age," which opens the first part of the volume called "Traditional City in Transition." Webber's thesis is that current discussions about the "crisis of the cities" have been clouded by the misconception that the geographically bounded city is still the relevant unit for discussion. He sees this confusion stemming "from the anachronistic thoughtways we have carried over from the passing era. We still have no adequate descriptive terms for the emerging social order," he continues, "and so we use, perforce, old labels that are no longer fitting. Because we have named them so, we suppose that the problems manifested *inside* [our italics] cities are, therefore and somehow, 'city problems.' Because societies in the past had been spatially and locally structured, and because urban societies used to be exclusively city-based, we seem still to assume that territoriality is a necessary attribute of social systems."

But this conceptual error, Webber points out, is a serious one, "leading us to seek local solutions to problems whose causes are not of local origin and hence not susceptible to municipal treatment. We have been tempted to apply city-building instruments to correct social disorders, and we have then been surprised to find that they do not work."

The theme of our lack of understanding of the real nature of urban problems persists, in one form or another, through all the papers. Edmund N. Bacon in "Urban Process," in "Part II: Processes and Goals for Change," opens baldly with the statement: "The failure of our cities is an intellectual one. It is brought about by the failure of the intellectuals to generate a viable concept of a modern city and a modern region." To attack the ignorance, Franklin A. Lindsay in "Managerial Innovation and the Cities" calls for, among other goals, greatly increased research on urban problems: technical, managerial, social, and economic. What is lacking now, he argues, "is a sense of direction and urgency. The most critical 'missing factors' are an understanding and acceptance of what is required to make headway on the complex interrelated social and economic problems, and the will to mobilize the necessary human and physical resources with the urgency that the problems demand." Lindsay holds forth a minimum goal to be reached ten years from now as a "level of R&D funding equal to one percent of the operating budgets of the nation's cities. The inadequacy of current spending on research is pointed out by a recent report of the Arden House Conference